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Note

Cycles of length 1 modulo 3 in graph ☆

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Abstract

We prove a conjecture of Saito that if a graph G with $\delta \geq 3$ has no cycle of length 1 (mod 3), then G has an induced subgraph which is isomorphic to the Petersen graph. The above result strengthened the result by Dean et al. that every 2-connected graph with $\delta \geq 3$ has a (1 mod 3)-cycle if G is not isomorphic to the Petersen graph.

Keywords: Cycle; Modulo; Chord

1. Introduction

We use Bondy and Murty [1] for terminology and notation not defined here and consider simple graphs only. Let G be a graph. If $x \in V(G)$ and $A \subseteq V(G)$, then $N_A(x)$ is the neighborhood of x in A. When $A = V(G)$, we denote $N_G(x) = N(x)$.

Let $C = v_1v_2 \cdots v_tv_1$ be a cycle of G. For $v_i \in V(C)$, we use v_i^+, v_i^- to denote the successor and predecessor of v_i on C, respectively. If $v_i, v_j \in V(C)$, then we use v_iCv_j or $v_j\bar{C}v_i$ to denote the v_i, v_j-arc of C with the same or opposite orientation with respect to the orientation of C and we will consider v_iCv_j and $v_j\bar{C}v_i$ both as paths and as vertex sets. If $e = v_iv_j \in E(G) \setminus E(C)$, then e is said to be a chord of C. Let $f = v_lv_s$ be another chord of C. If $v_l = v_i^+$ and $v_s \in v_jCv_i$ then e and f are said to be cross chords of C. Finally, we use $l(C)$ and $l(v_iCv_j)$ to denote the length of C and v_iCv_j, respectively.

For integers a and b a cycle is said to be an $(a \bmod b)$-cycle if its length is a (mod b). In [2], the following theorem is proved.

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Theorem 1. (Dean et al. [2]). *Let G be a 2-connected graph of minimum degree at least 3.*

(1) *If G is not isomorphic to* K_4 or $K_{3,n}$ *for some* $n \geqslant 3$, *then G has a* $(2\,mod\,3)$*-cycle.*

(2) *If G is not isomorphic to the Petersen graph, then G has a* $(1\,mod\,3)$*-cycle.*

In [3], Saito strengthened Theorem 1(1) in the following way.

Theorem 2. (Saito [3]). *Let G be a graph of minimum degree at least* 3. *If G has no* $(2\,mod\,3)$ *-cycle, then G contains either* K_4 *or* $K_{3,3}$ *as an induced subgraph.*

He also conjectured that, under the same assumption, G contains $(1 \bmod 3)$-cycle unless G contains the Petersen graph as an induced subgraph. In this paper, we prove this conjecture.

Theorem 3. *Let G be a graph of* $\delta \geq 3$. *If G has no* $(1\,mod\,3)$*-cycle, then G contains the Petersen graph as an induced subgraph.*

From this theorem, we have:

Corollary. *Every* 3*-regular connected graph except for the Petersen graph has a* $(1\,mod\,3)$*-cycle.*

2. Some lemma

In this section, assume that G has no cycle of length $1\,(\mathrm{mod}\,3)$, C a cycle in G and uv a chord on C.

Lemma 1.

(1) *If* $l(C) \equiv 0\,(\mathrm{mod}\,3)$, *then* $l(uCv) \equiv 1\,(\mathrm{mod}\,3)$ *and* $l(vCu) \equiv 2\,(\mathrm{mod}\,3)$ *or* $l(uCv) \equiv 2\,(\mathrm{mod}\,3)$ *and* $l(vCu) \equiv 1\,(\mathrm{mod}\,3)$.

(2) *If* $l(C) \equiv 2\,(\mathrm{mod}\,3)$, *then* $l(uCv) \equiv l(vCu) \equiv 1\,(\mathrm{mod}\,3)$.

Lemma 2. *Let* $l(C) \equiv 2\,(\mathrm{mod}\,3)$ *and* $u_1 \in uCv$. If $l(uCu_1) \equiv 1\,(\mathrm{mod}\,3)$ *or* $l(uCu_1) \equiv 0(\,\mathrm{mod}\,3)$, *then* $N_C(u_1) \subseteq uCv$.

Proof. Suppose there exists $v_1 \in vCu$ s.t. $u_1v_1 \in E(G)$. Then by Lemma 1 and $l(uCu_1) \equiv 1\,(\mathrm{mod}\,3)$, $l(u_1Cv) \equiv l(v_1Cu) \equiv 0\,(\mathrm{mod}\,3)$. Thus $l(vCv_1) \equiv 1\,(\mathrm{mod}\,3)$. But the cycle $uvCv_1u_1\bar{C}u$ would be a contradiction. The proof of $l(uCu_1) \equiv 0\,(\mathrm{mod}\,3)$ is similar. $\square$

Lemma 3. *Let* $l(C) \equiv 0\,(\mathrm{mod}\,3)$, $l(uCv) \equiv 2\,(\mathrm{mod}\,3)$ *and* $u_1 \in uCv$. *If* $l(uCu_1) \equiv 1\,(\mathrm{mod}\,3)$, *then* $N_C(u_1) \subseteq uCv$. *Specially,* $N_C(u^+) \subseteq uCv$ *and* $N_C(v^-) \subseteq uCv$.

Proof. Obviously, $l(u_1Cv) \equiv 1\,(\mathrm{mod}\,3)$. If there exists $v_1 \in vCu$ such that $u_1v_1 \in E(G)$, then we can assume that $l(v_1Cu_1) \equiv 2\,(\mathrm{mod}\,3)$ by Lemma 1. Thus $l(v_1Cu) \equiv 1\,(\mathrm{mod}\,3)$. But the cycle $v_1Cuv\bar{C}u_1v_1$ would be a contradiction. $\square$

Lemma 4. *Let* $l(C) \equiv 0\,(\mathrm{mod}\,3)$ *and* $l(uCv) \equiv 1\,(\mathrm{mod}\,3)$. *If there exists a chord* u_1v_1 *in* uCv, *then* $N_C(u_1^+) \subseteq u_1Cv_1$ *and* $N_C(v_1^-) \subseteq u_1Cv_1$.

Proof. Since $uCvu$ is a cycle of length 2 (mod 3), $N_C(u_1^+) \cap (uCu_1^- \cup v_1^+Cv) = N_C(v_1^-) \cap (uCu_1^- \cup v_1^+Cv) = \emptyset$ by Lemma 2. Suppose there exist $y_1, y_2 \in vCu$ so that $y_1u_1^+, y_2v_1^- \in E(G)$. We would consider three cases.

(1) $l(uCu_1) \equiv 0\,(\mathrm{mod}\,3)$ and then $l(v_1Cv) \equiv 0\,(\mathrm{mod}\,3)$. Then the cycles $y_1Cu_1^+y_1$ and $vCy_1u_1^+\bar{C}uv$ derive $l(y_1Cu) \not\equiv 2\,(\mathrm{mod}\,3)$ and $l(vCy_1) \not\equiv 1\,(\mathrm{mod}\,3)$. And the cycles $v_1^-Cy_2v_1^-$ and $y_2Cu_1v_1v_1^-y_2$ derive $l(vCy_2) \not\equiv 2\,(\mathrm{mod}\,3)$ and $l(y_2Cu) \not\equiv 1\,(\mathrm{mod}3)$. Since $l(vCu) \equiv 2\,(\mathrm{mod}\,3)$, we have that $l(y_1Cu) \equiv 0\,(\mathrm{mod}\,3)$ or $l(vCy_2) \equiv 0\,(\mathrm{mod}\,3)$. But the cycles $y_1Cuv\bar{C}v_1u_1u_1^+y_1$ or $vCy_2v_1^-v_1u_1\bar{C}uv$ would be a contradiction.

(2) $l(uCu_1) \equiv 1\,(\mathrm{mod}\,3)$ and then $l(v_1Cv) \equiv 2\,(\mathrm{mod}\,3)$. Then the cycle $y_1Cu_1^+y_1$ and $v_1Cy_1u_1^+u_1v_1$ derive $l(y_1Cu) \not\equiv 1\,(\mathrm{mod}\,3)$ and $l(vCy_1) \not\equiv 2\,(\mathrm{mod}\,3)$. The cycles $v_1^-Cy_2v_1^-$ and $y_2Cu_1v_1v_1^-y_2$ derive $l(vCy_2) \not\equiv 0\,(\mathrm{mod}\,3)$ and $l(y_2Cu) \not\equiv 0\,(\mathrm{mod}\,3)$. Since $l(vCu) \equiv 2\,(\mathrm{mod}\,3)$, we have $l(vCy_1) \equiv 0\,(\mathrm{mod}\,3)$ or $l(vCy_2) \equiv 1\,(\mathrm{mod}\,3)$. But the cycles $vCy_1u_1^+u_1\bar{C}uv$ or $vCy_2v_1^-\bar{C}uv$ would be a contradiction.

(3) $l(uCu_1) \equiv 2\,(\mathrm{mod}\,3)$ and then $l(v_1Cv) \equiv 1\,(\mathrm{mod}\,3)$. This situation is similar to (2).

From (1) – (3), $N_C(u_1^+) \subseteq u_1Cv_1$ and $N_C(v_1^-) \subseteq u_1Cv_1$. $\square$

3. Proof of Theorem 3

Assume that G has no cycle of length 1 (mod 3). Let $\mathscr{P}$ be the set of longest paths in G. For $P = x_1x_2\cdots x_n \in \mathscr{P}$, let $m(P) = max\{l : x_l \in N(x_1) \cap V(P)\}$. Choose $P \in \mathscr{P}$ so that $m(P)$ is as large as possible. Let $m(P) = m$, then $m > 4$ and $C = x_1\cdots x_mx_1$ is a cycle. Choose x_t such that x_t is the first vertex in $x_2^+Px_m$ with $x_1x_t \in E(G)$. Then x_1x_t is a chord of C. In the following proof, we would use the Search Procedure $S_1(x, y)$ on C.

Search Procedure $S_1(x, y)$ $(x, y \in V(C))$:

Step 1. Let $u_0{:=}x$ and $v_0{:=}y$; $i{:=}1$.

Step 2. $u_i{:=}v_{i-1}^-$ $(i \equiv 1\,(\mathrm{mod}\,2))$ or $u_i{:=}v_{i-1}^+$ $(i \equiv 0\,(\mathrm{mod}\,2))$. If $d_C(u_i) = 2$ or $N(u_i) \setminus u_{i-1}Cv_{i-1} \neq \emptyset$ $(i \equiv 1\,(\mathrm{mod}\,2))$ or $N(u_i) \setminus v_{i-1}Cu_{i-1} \neq \emptyset$ $(i \equiv 0\,(\mathrm{mod}\,2))$, then stop; else $v_i \in N(u_i) \setminus \{u_i^+, u_i^-\}$, $i{:=}i+1$ and goto Step 2.

Claim 1. *$l(C) \equiv 0 \pmod 3$ and then $l(x_1Px_m) \equiv 2 \pmod 3$.*

Proof. Suppose $l(C) \equiv 2 \pmod 3$. Then by Lemma 2 and the choice of P, $S_1(x_1, x_t)$ would stop at a vertex u_i such that $d(u_i) = 2$, a contradiction. □

Claim 2. *Assume, without loss of generality, that $l(x_1Cx_t) \equiv 1 \pmod 3$.*

Proof. If $l(x_1Cx_t) \equiv 2 \pmod 3$ then replace P by $P' = x_{m-1}\bar{P}x_1x_mPx_n$ and assume x' is the first vertex in $x_{m-3}\bar{P}x_m$ with $x'x_{m-1} \in E(G)$. If $x' \in x_tPx_{m-3}$ then $l(x_{m-1}P'x') \equiv 1 \pmod 3$ by Lemma 1. If $x' \in x_1Px_t$ and $l(x_{m-1}P'x') \equiv 2 \pmod 3$, then $l(x'Px_t) \equiv 0 \pmod 3$ by $l(x_tPx_{m-1}) \equiv 2 \pmod 3$. But the cycle $x'Px_tx_1x_mx_{m-1}x'$ would be a contradiction. Hence $l(x_{m-1}P'x') \equiv 1 \pmod 3$.

Claim 3. *There exists $x_s \in x_{t+1}Cx_m$ such that $x_sx_{t-1} \in E(G)$ and $l(x_tCx_s) \equiv 0 \pmod 3$.*

Proof. Obviously, $N(x_{t-1}) \subseteq V(C)$. If $N(x_{t-1}) \subseteq x_1Cx_t$, assume $y \in N(x_{t-1}) \cap x_1Cx_t$, then by Lemma 4 and the choice of P, $S_1(x_{t-1}, y)$ would stop at u_i with $d(u_i) = 2$, a contradiction. Hence there exists $x_s \in x_{t+1}Cx_m$ such that $x_sx_{t-1} \in E(G)$. Obviously, $l(x_sCx_1) \not\equiv 1 \pmod 3$ and $l(x_tCx_s) \not\equiv 2 \pmod 3$. But $l(x_tCx_1) \equiv 2 \pmod 3$, so $l(x_tCx_s) \equiv 0 \pmod 3$ and $l(x_sCx_1) \equiv 2 \pmod 3$.

Claim 4. *There exists $x_l \in x_2Cx_{t-1}$ such that $x_lx_{s-1} \in E(G)$ and $l(x_1Cx_l) \equiv 1 \pmod 3$.*

Proof. By the choice of P, $N(x_{s-1}) \subseteq V(C)$. Since $l(x_{t-1}Cx_s) \equiv 1 \pmod 3$, there exists $x_l \in x_{t-2}\bar{C}x_{s+1}$ such that $x_lx_{s-1} \in E(G)$ by Claim 3. If $x_l \in x_{s+1}Cx_1$, then $l(x_sCx_l) \not\equiv 2 \pmod 3$ and the cycle $x_lCx_{t-1}x_sx_{s-1}x_l$ derives $l(x_lCx_1) \not\equiv 1 \pmod 3$. Hence $l(x_sCx_l) \equiv 0 \pmod 3$, $l(x_lCx_1) \equiv 2 \pmod 3$ and $x_l \neq x_1$. But the cycle $x_lCx_1x_tx_{t-1}x_sx_{s-1}x_l$ would be a contradiction. Therefore $x_l \in x_2Cx_{t-1}$. Let x_l be the first vertex in x_2Cx_{t-1} with $x_lx_{s-1} \in E(G)$. By Lemma 3, $l(x_{s-1}Cx_l) \equiv 1 \pmod 3$ and $l(x_1Cx_l) \equiv 1 \pmod 3$.

Claim 5. $s = m - 1$.

Proof. Suppose $s \neq m-1$, then $N(x_{m-1}) \subseteq V(C)$ by the choices of P. Denote $y \in N(x_{m-1}) \backslash \{x_m, x_{m-2}\}$. If $y \in x_tCx_{s-1}$ thenthe cycles $x_tCyx_{m-1}x_mx_1x_t$ and $x_{t-1}Cyx_{m-1}\bar{C}x_sx_{t-1}$ derive $l(x_tCy) \not\equiv 0 \pmod 3$ and $l(x_tCy) \not\equiv 1 \pmod 3$. So $l(x_tCy) \equiv 2 \pmod 3$ and then $l(yCx_s) \equiv 1 \pmod 3$. But the cycle $yCx_sx_{t-1}x_tx_1x_mx_{m-1}y$ would be a contradiction.

If $y \in x_1Cx_l$, then $l(x_{m-1}Cy) \equiv 1 \pmod 3$ by the cycle $x_{s-1}Cx_lx_{s-1}$ with length 2 (mod 3). By Claim 4, $l(x_1Cx_l) \equiv 1 \pmod 3$ and then $l(yCx_l) \equiv 2 \pmod 3$. But the cycle $yCx_lx_{s-1}x_sx_{t-1}x_tx_1x_mx_{m-1}y$ would be a contradiction.

If $y \in x_l^+Cx_{t-1}$, then the cycles $x_lCyx_{m-1}\bar{C}x_{s-1}x_l$ and $yCx_{t-1}x_sCx_{m-1}y$ derive $l(x_lCy) \not\equiv 1 \pmod 3$ and $l(yCx_{t-1}) \not\equiv 2 \pmod 3$, respectively. Since $l(x_lCx_{t-1}) \equiv$

$2\,(\text{mod}\,3)$, $l(x_lCy) \equiv 2\,(\text{mod}\,3)$ and $l(yCx_{t-1}) \equiv 0\,(\text{mod}\,3)$. But the cycle $yCx_{t-1}x_sx_{s-1}x_l\bar{C}x_{m-1}y$ would be a contradiction.

Therefore $y \in x_sCx_m$. Since $l(x_{s-1}Cx_l) \equiv 1\,(\text{mod}\,3)$, $N(y^+) \subseteq yCx_{m-1}$ by Lemma 4 and the choice of P. Thus $S_1(x_{m-1}, y)$ would stop at u_i with $d(u_i) = 2$, a contradiction. $\square$

Claim 6. $x_l = x_2$.

Proof. Suppose $x_l \neq x_2$, then $N(x_2) \subseteq V(C)$. Let $y \in N(x_2) \setminus \{x_1, x_3\}$. Obviously, $y \notin \{x_{m-1}, x_t, x_{t-1}\}$ and $y \neq x_m$ by Lemma 2. If $y \in x_l^+Cx_{t-1}$, then $l(x_2Cy) \equiv 1\,(\text{mod}\,3)$ by Lemma 1. Thus $l(x_lCy) \equiv 1\,(\text{mod}\,3)$. But the cycle $x_lCyx_2x_1x_mx_{m-1}x_{m-2}x_l$ would be a contradiction.

If $y \in x_tCx_{m-2}$, then the cycles $yx_2\bar{C}y$ and $x_tCyx_2x_1x_t$ derive $l(yCx_{m-2}) \not\equiv 2\,(\text{mod}\,3)$ and $l(x_tCy) \not\equiv 1\,(\text{mod}\,3)$. Since $l(x_tCx_{m-2}) \equiv 2\,(\text{mod}\,3)$, $l(x_tCy) \equiv 2\,(\text{mod}\,3)$ and $l(yCx_{m-2}) \equiv 0\,(\text{mod}\,3)$. But the cycle $x_l\bar{C}x_2y\bar{C}x_{t-1}x_{m-1}x_{m-2}x_l$ would be a contradiction.

Hence $y \in x_3Cx_l$. So $S_1(x_2, y)$ would stop at u_i with $d(u_i) = 2$ by Lemma 4, a contradiction.

Claim 7. $d(x_1) = d(x_{m-1}) = d(x_2) = d(x_{m-2}) = 3$.

Proof. Suppose there exists $y \in V(C) \setminus \{x_m, x_2\}$ such that $yx_1 \in E(G)$, then $y \in x_{t+1}Cx_{m-1}$ by the choice of x_t. Obviously, $y \neq x_{m-2}$ and $y \neq x_{m-1}$. On the other hand, the cycles yCx_1y and $x_tCyx_1x_2x_{m-2}x_{m-1}x_{t-1}x_t$ derive $l(yCx_{m-2}) \not\equiv 0\,(\text{mod}\,3)$ and $l(x_tCy) \not\equiv 1\,(\text{mod}\,3)$. Since $l(x_tCx_{m-2}) \equiv 2\,(\text{mod}\,3)$, $l(yCx_{m-2}) \equiv 2\,(\text{mod}\,3)$. But the cycle $yCx_{m-1}x_{t-1}x_tx_1y$ would be a contradiction. Hence $d(x_1) = 3$.

Suppose there exists $y' \in V(C) \setminus \{x_1, x_3\}$ such that $y'x_2 \in E(G)$, then $y' \notin \{x_{m-1}, x_m\}$. If $y' \in x_3Cx_t$, then $S_1(x_2, y')$ would stop at u_i with $d(u_i) = 2$ by the choice of P and Lemma 4, a contradiction. So $y' \in x_tCx_{m-3}$. Since the cycles $y'Cx_{m-2}x_2y'$ and $x_tCy'x_2x_1x_t$ derive $l(y'Cx_{m-2}) \not\equiv 2\,(\text{mod}\,3)$ and $l(x_tCy') \not\equiv 1\,(\text{mod}\,3)$, $l(x_tCy') \equiv 2\,(\text{mod}\,3)$. But the cycle $x_{t-1}Cy'x_2x_{m-2}x_{m-1}x_{t-1}$ would be a contradiction. Hence $d(x_2) = 3$.

By the proof of Claims 5 and 6, $d(x_{m-1}) = d(x_{m-2}) = 3$. $\square$

Claim 8. $N(x_3) \subseteq V(P)$.

Proof. Suppose there exists $y_0 \notin V(P)$ s.t. $x_3y_0 \in E(G)$, then $N(y_0) \subseteq V(C)$ by the choice of P. Let $y_1 \in N(y_0) \setminus \{x_3\}$, then $y_1 \neq x_t$. If $y_1 \in x_4Cx_{t-1}$, then the cycles $x_3Cy_1y_0x_3$ and $x_3y_0y_1Cx_tx_1x_2x_3$ force $l(x_3Cy_1) \equiv 1\,(\text{mod}\,3)$. But when $l(x_3Cy_1) \equiv 1\,(\text{mod}\,3)$, we have $l(y_1Cx_3) \equiv 2\,(\text{mod}\,3)$ and then the cycle $y_1Cx_3y_0y_1$ would be a contradiction. If there exist $y_1, y_2 \in x_{t+1}Cx_{m-3}$, say $y_2 \in y_1^+Cx_{m-3}$, such that $y_1, y_2 \in N(y_0)$, then the cycles $y_0y_1Cy_2y_0$ and $y_0y_2Cy_1y_0$ derive $l(y_1Cy_2) \not\equiv 2\,(\text{mod}\,3)$ and

$l(y_2Cy_1) \not\equiv 2\,(\mathrm{mod}\,3)$. Since $l(C) \equiv 0\,(\mathrm{mod}\,3)$, $l(y_2Cy_1) \equiv 0\,(\mathrm{mod}\,3)$. But the cycle $y_0y_2Cy_1y_0$ has a chords x_1x_t, x_2x_{m-2}, a contradiction with Lemma 2. Hence $|N(y_0) \cap x_{t+1}Cx_{m-3}| \leq 1$. Since $\delta \geq 3$, $x_my_0 \in E(G)$. Obviously, $t \geq 5$. Then the path $x_4Px_{t-1}x_{m-1}\bar{P}x_tx_1x_2x_3y_0x_mPx_n$ would be longer than P, a contradiction. □

Claim 9. *There exists $x_k \in x_{m+1}Px_n$ such that $x_kx_3 \in E(G)$ and $l(x_mPx_k) \equiv 1\,(\mathrm{mod}\,3)$.*

Proof. Suppose that $N(x_3) \subseteq V(C)$. Since $l(x_{m-1}Cx_{t-1}) \equiv 2\,(\mathrm{mod}\,3)$, $N(x_3) \subseteq x_{m-1}Cx_{t-1}$ by Lemma 3. By Claim 7 and the fact $x_3x_m \notin E(G)$, $N(x_3) \subseteq x_2Cx_{t-1}$. Denote $x_3' \in N(x_3) \setminus \{x_2, x_4\}$, then $l(x_3Cx_3') \equiv 1\,(\mathrm{mod}\,3)$. Thus $S_1(x_3, x_3')$ would stop at $N(u_{2i+1}) \setminus (u_{2i}Cv_{2i}) \neq \emptyset$ or $N(u_{2i}) \setminus v_{2i-1}Cu_{2i-1} \neq \emptyset$. Assume $N(u_{2i+1}) \setminus \{u_{2i}Cv_{2i}\} \neq \emptyset$, then by Lemma 4, we just have $N(u_{2i+1}) \cap x_{m+1}Px_n \neq \emptyset$ or $N(u_{2i+1}) \setminus V(P) \neq \emptyset$. We use induction to prove $l(x_3Cu_k) \equiv 0\,(\mathrm{mod}\,3)$ for $k = 1, 2, \ldots$. Since $l(x_3Cx_3') \equiv 1\,(\mathrm{mod}\,3)$, $l(x_3Cu_1) \equiv 0\,(\mathrm{mod}\,3)$. Suppose it is true for all integers less than k. Assume k is an even number. Since $l(x_3Cx_3'x_3) \equiv 2\,(\mathrm{mod}\,3)$, $l(v_{k-1}Cu_{k-1}) \equiv 1\,(\mathrm{mod}\,3)$ by Lemma 1. Hence $l(u_kCu_{k-1}) \equiv 0\,(\mathrm{mod}\,3)$ and then $l(x_3Cu_k) \equiv 0\,(\mathrm{mod}\,3)$. Specially, $l(x_3Cu_{2i+1}) \equiv 0\,(\mathrm{mod}\,3)$.

If there exists $v \in N(u_{2i+1}) \cap x_{m+1}Px_n$, then the cycles $x_mCu_{2i+1}v\bar{P}x_m$ and $u_{2i+1}Cx_3'x_3x_2x_1x_mPvu_{2i+1}$ derive $l(x_mPv) \not\equiv 0\,(\mathrm{mod}\,3)$ and $l(x_mPv) \not\equiv 1\,(\mathrm{mod}\,3)$. But when $l(x_mPv) \equiv 2\,(\mathrm{mod}\,3)$, the cycle $u_{2i+1}\bar{C}x_3x_2x_{m-2}x_{m-1}x_mPvu_{2i+1}$ would be a contradiction.

If there exists $y_0 \notin V(P)$ such that $y_0u_{2i+1} \in E(G)$, then by the choice of P, $N(y_0) \subseteq V(C)$. If there exists $y_1 \in N(y_0) \cap x_2Cx_{t-1}$ and $y_1 \neq u_{2i+1}$, assume $y_1 \in u_{2i+1}Cx_{t-1}$, then the cycles $u_{2i+1}Cy_1y_0u_{2i+1}$ and $y_1Cu_{2i+1}y_0y_1$ derive $l(u_{2i+1}Cy_1) \not\equiv 2\,(\mathrm{mod}\,3)$ and $l(y_1Cu_{2i+1}) \not\equiv 2\,(\mathrm{mod}\,3)$. Since $l(C) \equiv 0\,(\mathrm{mod}\,3)$, $l(y_1Cu_{2i+1}) \equiv 0\,(\mathrm{mod}\,3)$. But the cycle $y_1Cu_{2i+1}y_0y_1$ has a cross chords x_1x_t and x_2x_{m-2}, a contradiction with Lemma 2. Hence $N(y_0) \cap x_2Cx_{t-1} = \{u_{2i+1}\}$ and similarly, $|N(y_0) \cap x_tCx_{m-2}| \leq 1$. Hence $x_mu_{2i+1} \in E(G)$ and the cycle $u_{2i+1}Cx_3'x_3x_2x_1x_my_0u_{2i+1}$ would be a contradiction.

Hence there exists $x_k \in x_{m+1}Px_n$ s.t. $x_3x_k \in E(G)$. Obviously, the cycles $x_mPx_kx_3x_2x_1x_m$ and $x_mPx_kx_3Cx_tx_1x_m$ derive $l(x_mPx_k) \equiv 1\,(\mathrm{mod}\,3)$. □

Claim 10. $t = 5$.

Proof. Suppose $t > 5$, then x_{t-2} exists. If $\exists v_0 \notin V(P)$ s.t. $x_{t-2}v_0 \in E(G)$, then $N(v_0) \subseteq V(C)$ by the choice of P. Obviously, $N(v_0) \cap x_2Cx_{t-1} = \{x_{t-2}\}$ and $|N(v_0) \cap x_tCx_{m-2}| \leq 1$. Hence $x_mv_0 \in E(G)$. But the cycle $x_3Cx_{t-2}v_0x_mPx_kx_3$ would be a contradiction. Thus $N(x_{t-2}) \subseteq V(P)$.

Let $y \in N(x_{t-2}) \setminus \{x_{t-1}, x_{t-3}\}$, then $y \neq x_{m-1}$. Since $l(x_{m-1}Cx_{t-1}) \equiv 2\,(\mathrm{mod}\,3)$, $N_C(x_{t-2}) \subseteq x_mCx_{t-1}$ by Lemma 3. If $y \in x_mPx_n$ then the cycles $x_1x_mPyx_{t-2}x_{t-1}x_tx_1$ and $x_1x_mPyx_{t-2}\bar{C}x_1$ derive $l(x_mPy) \equiv 1\,(\mathrm{mod}\,3)$. But the cycles $x_{t-2}y\bar{P}x_kx_3x_2x_{m-2}x_{m-1}x_{t-1}x_{t-2}$ $(x_k \in x_mPy)$ and $x_{t-2}yPx_kx_3x_2x_{m-2}x_{m-1}x_{t-1}x_{t-2}$ $(y \in x_mPx_k)$ would be a contradiction. Therefore $y \in x_2Cx_{t-2}$. Then by the same proof as that of Claim 9, we can derive a contradiction.

Claim 11. $m=9$.

Proof. By Claim 10 and replacing P with $x_{m-1}\bar{P}x_1x_mPx_n$, we have $m=9$ and there exists $x_h \in x_mPx_n$ such that $x_6x_h \in E(G)$ and $l(x_mPx_h) \equiv 1 \pmod 3$.

By Claims 7, 10, 11 and the choice of P, we have

Note 1 $d(x_1)=d(x_2)=d(x_4)=d(x_5)=d(x_7)=d(x_8)=3$.

By the proofs of Claims 9 and 11, we have

Note 2 *For any x_9-x_k path $P_1 \subseteq x_mPx_n$* and x_9-x_h path $P_2 \subseteq x_mPx_n$, $l(P_1) \equiv l(P_2) \equiv 1 \pmod 3$.

Claim 12. $k=10$.

Proof. Suppose that $k>10$. We first use the following Search Procedure (2) to get a x_9-x_k path.

Search Procedure $S_2(x,y)$ $(x,y \in V(P))$:

Step 1. Let $a_0{:=}x$, $b_0{:=}y$; $i{:=}0$.

Step 2. Let $a_i{:=}b_{i-1}^{-}$ (on P). If $N(a_i)\cap b_{i-1}^{+}Px_n=\emptyset$, then stop; else let $b_i{:=}N(a_i)\cap b_{i-1}^{+}Px_n$, $i{:=}i+1$ and goto step 2.

Suppose that $S_2(x_9,x_k)$ stop at i_0+1 (assume, $i_0 \equiv 1 \pmod 2$), then we have a x_9-x_k path $Q{=}x_9Pa_1b_1P\cdots a_{i_0}b_{i_0}\bar{P}b_{i_0-1}a_{i_0-1}\cdots b_2a_2\bar{P}x_k$. By Note 2, $l(Q)\equiv 1 \pmod 3$. Thus we have a cycle $C'=x_1x_9Qx_kx_3x_2x_7x_6x_5x_1$ with length $2 \pmod 3$.

P(1) *For any $x \in V(Q)$, if there exists $y \notin V(Q)$ so that $xy \in E(G)$, then $|N(y)\cap(V(C)\cup V(Q))| \le 2$.*

Let $y_0 \in N(y)\cap(V(C)\cup Q)$, then $y_0 \in \{x_6\}\cup Q$ by Claim 8 and Note 1. If $y_0 \in Q\setminus\{x\}$, say $y_0 \in x_9Qx$, then $l(y_0Qx) \not\equiv 0 \pmod 3$ and $l(y_0Qx)\not\equiv 2 \pmod 3$ by $l(C')\equiv 2 \pmod 3$. Thus $l(y_0Qx)\equiv 1 \pmod 3$ and then we have a x_9-x_k path $x_9Qy_0yxQx_k$ which has length $2 \pmod 3$, a contradiction with Note 2. So $N(y)\cap(V(C)\cup Q) \subseteq \{x_6,x\}$.

Next we would consider a_{i_0+1} and would show $N(a_{i_0+1}) \subseteq b_{i_0}C'b_{i_0-1}$, that is, $N(a_{i_0+1}) \subseteq b_{i_0-1}Pb_{i_0}$.

If there exists $y \notin V(P)$ such that $ya_{i_0+1} \in E(G)$, then $N(y) \subseteq x_1Pb_{i_0}$ by the choice of P. Thus $d(y)\le 2$ by P(1), a contradiction. Hence $N(a_{i_0+1}) \subseteq x_1Pb_{i_0}$. When $i_0 \ge 3$, we have a parallel chords $a_{i_0}b_{i_0-1}$, $b_{i_0-2}a_{i_0-1}$ in C' and $l(C')\equiv 2 \pmod 3$ and $l(b_{i_0-2}C'a_{i_0+1}) \equiv 1 \pmod 3$, then $N(a_{i_0+1}) \subseteq b_{i_0-2}C'a_{i_0-1}$ by Lemma 2. Let $b_{i_0+1} \in N(a_{i_0+1})$ and b_{i_0+1} is not the successor or predecessor of a_{i_0+1} on C'. Obviously, $b_{i_0+1} \ne a_{i_0-1}, a_{i_0}$. If $b_{i_0+1} \in b_{i_0-2}C'a_{i_0}$, then the cycles $b_{i_0+1}C'a_{i_0+1}b_{i_0+1}$ and $b_{i_0+1}a_{i_0+1}C'a_{i_0-1}b_{i_0-2}C'b_{i_0+1}$ derive $l(b_{i_0+1}C'a_{i_0})\not\equiv 1 \pmod 3$ and $l(b_{i_0-2}C'b_{i_0+1}) \not\equiv 2 \pmod 3$. By $l(b_{i_0-2}C'a_{i_0})\equiv 2 \pmod 3$, $l(b_{i_0-2}C'b_{i_0+1})\equiv 0 \pmod 3$. Since $l(x_9C'b_{i_0-2}) + l(a_{i_0-1}C'x_k) \equiv 0 \pmod 3$, the x_9-x_k path $x_9C'b_{i_0+1}a_{i_0+1}b_{i_0}a_{i_0}b_{i_0-1}C'x_k$ would contradict with Note 2. Therefore $N(a_{i_0+1}) \subseteq b_{i_0}C'b_{i_0-1}$. If $i_0=1$, we similarly have $N(a_2) \subseteq x_kPb_1$.

By $S_2(a_{i_0+1}, b_{i_0+1})$, we would have a vertex u_{j_0+1} with $N(u_{j_0+1}) \setminus V(C') \neq \emptyset$ by $l(C') \equiv 2\,(\mathrm{mod}\,3)$. By P(1) and the choice of P, $N(u_{j_0+1}) \subseteq V(P)$, thus $N(u_{j_0+1}) \subseteq b_{i_0}Px_n$ by Note 1. Let $v_{j_0+1} \in N(u_{j_0+1}) \cap b_{i_0}Px_n$, then we have a $x_9 - x_k$ path $Q' = x_9a_1b_1P\cdots a_{i_0}b_{i_0}Pv_{j_0+1}u_{j_0+1}\bar{P}u_{j_0}v_{j_0}P\cdots u_0v_0\bar{P}b_{i_0}a_{i_0}\bar{P}\cdots b_2a_2\bar{P}x_k$ (assume, $j_0 \equiv 1\,(\mathrm{mod} 2)$) with length $1\,(\mathrm{mod}\,3)$ by Note 2. Thus, the cycle $x_1x_9Q'x_kx_3x_2x_7x_6x_5x_1$ with length $2\,(\mathrm{mod}\,3)$ but has a cross chords $b_{i_0}u_0, v_0u_1$, a contradiction with Lemma 2. Hence $k = 10$. □

Similarly, we have:

Claim 13. $h = 10$.

Thus, $G[\{x_1, \cdots, x_{10}\}]$ is a Petersen graph. □

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